Lecture - 38.

Stoke's Theorem... Cont.

* To apply Stoke's Theorem, we need to calculate:

1. $\text{curl } \vec{V} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_1 & V_2 & V_3 \end{vmatrix}$

2. $\hat{n} = \dfrac{\text{grad } f}{|\text{grad } f|} = \dfrac{\vec{\nabla} f}{|\vec{\nabla} f|}$

3. $dA = \dfrac{dx\,dy}{\hat{n}\cdot\hat{k}}$ [Projecting on xy-plane]

Then, By Stoke's Thm:

$$\oint_C \vec{V}\cdot d\vec{r} = \iint_S (\text{curl } \vec{V})\cdot\hat{n}\, dA$$

Q: Evaluate $\oint_C \vec{V}\cdot d\vec{r}$ using Stoke's theorem where $\vec{V} = (x\hat{i} + y\hat{j} + z\hat{k})$, S is the region bounded by sphere $x^2+y^2+z^2=16$

Sol: Here $\vec{V} = (x\hat{i}+y\hat{j}+z\hat{k})$

Comparing with: $\vec{V} = (V_1\hat{i} + V_2\hat{j} + V_3\hat{k})$

$$\text{curl } \vec{V} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix}$$

$$= 0\hat{i} + 0\hat{j} + 0\hat{k} = \vec{0}$$

$\therefore$ By Stoke's Thm:

$$\oint_C \vec{V}\cdot d\vec{r} = \iint_S (\underbrace{\text{curl } \vec{V}}_{\vec{0}})\cdot\hat{n}\, dA = 0 \quad \text{Ans}$$

Lecture – 38.

Stoke's Theorem-- Continued:

Q: Evaluate $\oint_C \vec{V}\cdot d\vec{r}$ using Stoke's theorem, where $\vec{V} = (z\hat{i} + x\hat{j} + z\hat{k})$ and S is portion of the sphere $x^2+y^2+z^2=9$ above xy-plane.

Sol: Here $\vec{V} = (z\hat{i} + x\hat{j} + z\hat{k})$

$$\Rightarrow \text{curl}\,\vec{V} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z & x & z \end{vmatrix}$$

$$= \hat{i}(0-0) - \hat{j}(0-1) + \hat{k}(1-0)$$

$$\boxed{\text{curl}\,\vec{V} = (0\hat{i} + 1\hat{j} + 1\hat{k})}$$

Let $f = x^2+y^2+z^2 = 9$

$$\Rightarrow \vec{\nabla} f = (f_x\hat{i} + f_y\hat{j} + f_z\hat{k})$$

$$= (2x\hat{i} + 2y\hat{j} + 2z\hat{k})$$

$$\Rightarrow \hat{n} = \frac{\vec{\nabla} f}{|\vec{\nabla} f|} = \frac{(2x\hat{i} + 2y\hat{j} + 2z\hat{k})}{\sqrt{(2x)^2 + (2y)^2 + (2z)^2}}$$

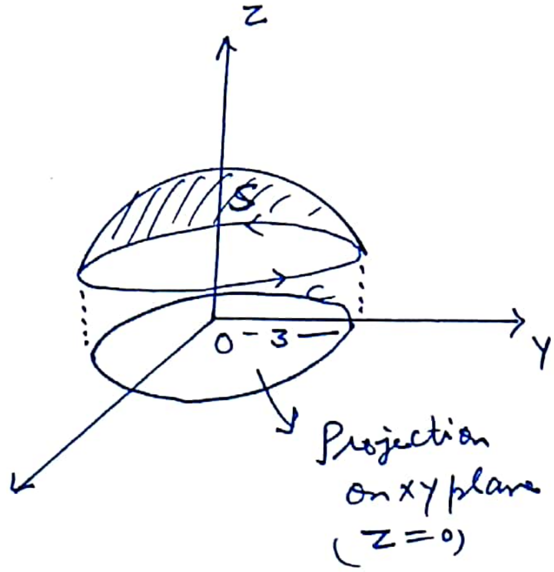
$$= \frac{\cancel{2}(x\hat{i} + y\hat{j} + z\hat{k})}{\cancel{2}\sqrt{x^2+y^2+z^2}}$$

$$\Rightarrow \boxed{\hat{n} = \frac{(x\hat{i} + y\hat{j} + z\hat{k})}{3}}$$

Let us take projection on XY-plane
$\downarrow$
$(Z=0)$

$\therefore \quad S: \; x^2+y^2+0 = 9$

$\Rightarrow \quad x^2+y^2 = 9$
$\downarrow$
It is a circle of rad. 3



Now $\quad dA = \dfrac{dx\,dy}{\hat{n}\cdot\hat{k}}$

$$\Rightarrow \quad dA = \frac{dx\,dy}{\frac{(x\hat{i}+y\hat{j}+z\hat{k})}{3}\cdot(0\hat{i}+\hat{j}+\hat{k})}$$

$$= \frac{dx\,dy}{\frac{(y+z)}{3}}$$

$$\Rightarrow \quad \boxed{dA = \frac{3\,dx\,dy}{(y+z)}}$$

$\therefore$ By Stoke's Thm:

$$\oint_C \vec{V}\cdot d\vec{r} = \iint_S (\text{curl}\,\vec{V})\cdot\hat{n}\,dA$$

$$= \iint_S \left[(0\hat{i}+\hat{j}+\hat{k})\cdot\frac{(x\hat{i}+y\hat{j}+z\hat{k})}{3}\right]\frac{3\,dx\,dy}{(y+z)}$$

$$= \iint_S \frac{(y+z)}{3}\cdot\frac{3\,dx\,dy}{(y+z)}$$

$$\Rightarrow \oint_C \vec{V}\cdot d\vec{r} = \iint_S \frac{\cancel{(y+z)}}{\cancel{3}} \cdot \frac{\cancel{3}\, dx\, dy}{\cancel{(y+z)}}$$

$$= \iint_S dx\, dy$$

$$= (\text{Area of circle } x^2+y^2=9)$$

$$= \pi r^2$$

$$= \pi (3)^2$$

$$= 9\pi$$

$$\Rightarrow \boxed{\oint_C \vec{V}\cdot d\vec{r} = 9\pi}$$